**Normalized Perturbation Theory**

You will notice that the perturbative expansion for the eigenvectors does not preserve their normalization. In some applications, this is not good enough.

**Normalized Eigenfunctions**

A way out of this is to normalize the eigenvector after the perturbative expansion has been performed. Let’s go to second order:



For shorthand we have:



Normalizing:



Therefore the normalization is:



Dividing the wavefunction by this normalization we get:





Therefore up to second order in V, our normalized wavefunction is:



**Checking orthogonality of normalized eigenfunctions**

Are these eigenfunctions orthogonal? Well let’s just check,



which is:



which is:



Well continuing,



which is:



which is:



Now:



so this simplifies to:



as it should. Consider computing the normalization directly:



which is:



as required. Now consider…



Consider what |ψmn|2 is…



The modulus squared is:



Simplifying a little bit…



When sum over m, I do get the ψ’s to go away as they should. Continuing to simplify…



Could write this as:



Now if sum over m we would get:



What if sum over n?



and,



and last,



Continuing…



Splitting up the first sum by switching i and n in the second part I have:



Now combining terms, and I’m going to assume that all the terms are real, consistent with my point of application.



So at least for real elements, the normalization remains!

**Example**

Suppose we want to perturbatively expand the adjoint eigenvectors. For instance, suppose we have:



We can develop a perturbative expansion of |n′> that is normalized to order V2. What I would like to do, however, is develop a perturbative expansion of |m> w/r to the |n′> basis that is normalized to order V2. And υ† is:



Perhaps a test case. Consider the 2×2 Hamiltonian,



the eigenvectors and eigenvalues are…



Eigenvectors,



Normalizing,



So we have:



and this simplifies a bit to,



Now is this normalized across the rows? According to Mathcad, yes. Now let’s do perturbative expansion up to second order in V. Then we have:



Now observe that this is normalized across both columns and rows up to order V2 already. But this is because up to this order, the matrix is symmetric. This doesn’t have to happen. Let’s look at the eigenvectors…according to column normalized formulas,



and,



So that works too. The others work as well I imagine.